THE 1989 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let x_1, x_2, ..., x_n be positive real numbers, and let

$$S = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \cdots + \frac{S^n}{n!}.$$

Question 2

Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

Question 3

Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1$, 2, and 3, suppose that B_n is the midpoint of A_nA_{n+1}, and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n, and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

Question 4

Let S be a set consisting of m pairs (a, b) of positive integers with the property that $1 \leq a < b \leq n$. Show that there are at least

$$4m \cdot \frac{(m - \frac{n^2}{4})}{3n}$$

triples (a, b, c) such that (a, b), (a, c), and (b, c) belong to S.

Question 5

Determine all functions f from the reals to the reals for which

(1) $f(x)$ is strictly increasing,
(2) $f(x) + g(x) = 2x$ for all real x,

where $g(x)$ is the composition inverse function to $f(x)$. (Note: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)

THE 1991 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let G be the centroid of triangle ABC and M be the midpoint of BC. Let X be on AB and Y on AC such that the points X, Y, and G are collinear and XY and BC are parallel. Suppose that XC and GB intersect at Q and YB and GC intersect at P. Show that triangle MPQ is similar to triangle ABC.

Question 2

Suppose there are 997 points given in a plane. If every two points are joined by a line segment with its midpoint coloured in red, show that there are at least 1991 red points in the plane. Can you find a special case with exactly 1991 red points?

Question 3

Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{a_1 + a_2 + \cdots + a_n}{2}.$$

Question 4

During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.

Question 5

Given are two tangent circles and a point P on their common tangent perpendicular to the lines joining their centres. Construct with ruler and compass all the circles that are tangent to these two circles and pass through the point P.

THE 1992 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

A triangle with sides a, b, and c is given. Denote by s the semiperimeter, that is $s = (a+b+c)/2$. Construct a triangle with sides $s-a$, $s-b$, and $s-c$. This process is repeated until a triangle can no longer be constructed with the side lengths given.
For which original triangles can this process be repeated indefinitely?

Question 2

In a circle C with centre O and radius r, let C_1, C_2 be two circles with centres O_1, O_2 and radii r_1, r_2 respectively, so that each circle C_i is internally tangent to C at A_i and so that C_1, C_2 are externally tangent to each other at A.
Prove that the three lines OA, O_1A_2, and O_2A_1 are concurrent.

Question 3

Let n be an integer such that $n > 3$. Suppose that we choose three numbers from the set $\{1, 2, \ldots, n\}$. Using each of these three numbers only once and using addition, multiplication, and parenthesis, let us form all possible combinations.

(a) Show that if we choose all three numbers greater than $n/2$, then the values of these combinations are all distinct.

(b) Let p be a prime number such that $p \leq \sqrt{n}$. Show that the number of ways of choosing three numbers so that the smallest one is p and the values of the combinations are not all distinct is precisely the number of positive divisors of $p-1$.

Question 4

Determine all pairs (h, s) of positive integers with the following property:

If one draws h horizontal lines and another s lines which satisfy

(i) they are not horizontal,

(ii) no two of them are parallel,

(iii) no three of the $h+s$ lines are concurrent,

then the number of regions formed by these $h+s$ lines is 1992.

Question 5

Find a sequence of maximal length consisting of non-zero integers in which the sum of any seven consecutive terms is positive and that of any eleven consecutive terms is negative.

THE 1993 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral such that all sides have equal length and angle ABC is 60 deg. Let l be a line passing through D and not intersecting the quadrilateral (except at D). Let E and F be the points of intersection of l with AB and BC respectively. Let M be the point of intersection of CE and AF.
Prove that $CA^2 = CM \times CE$.

Question 2

Find the total number of different integer values the function

$$f(x) = [x] + [2x] + [\frac{5x}{3}] + [3x] + [4x]$$

takes for real numbers x with $0 \leq x \leq 100$.

Question 3

Let

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \quad \text{and} \\ g(x) &= c_{n+1} x^{n+1} + c_n x^n + \cdots + c_0 \end{aligned}$$

be non-zero polynomials with real coefficients such that $g(x) = (x + r)f(x)$ for some real number r. If $a = \max(|a_n|, \ldots, |a_0|)$ and $c = \max(|c_{n+1}|, \ldots, |c_0|)$, prove that $\frac{a}{c} \leq n + 1$.

Question 4

Determine all positive integers n for which the equation

$$x^n + (2 + x)^n + (2 - x)^n = 0$$

has an integer as a solution.

Question 5

Let P_1, P_2, $\ldots$, $P_{1993} = P_0$ be distinct points in the xy-plane with the following properties:

(i) both coordinates of P_i are integers, for $i = 1, 2, \ldots, 1993$;

(ii) there is no point other than P_i and P_{i+1} on the line segment joining P_i with P_{i+1} whose coordinates are both integers, for $i = 0, 1, \ldots, 1992$.

Prove that for some i, $0 \leq i \leq 1992$, there exists a point Q with coordinates (q_x, q_y) on the line segment joining P_i with P_{i+1} such that both $2q_x$ and $2q_y$ are odd integers.